X Junior Balkan Olympiad in Informatics

Skopje, 2016

Day 2: Exam

***Analysis***:

Let $a_1, a_2, a_3, a_4, a_5, a_6$ be one combination of points of one chosen set of questions, where $a_1 \le a_2 \le a_3 \le a_4 \le a_5 \le a_6$ .

Note that there are 2 possible cases:

1. $a_1 + a_4 = a_2 + a_3 = a_5 = a_6$ and
2. $a_1 + a_2 + a_3 = a_4 = a_5 = a_6$

Let $n_i$ be the number of questions being worth i points.

Next, we analyze how to calculate the number of ways of choosing a set of questions in $O(n^2)$ time complexity. Let us consider the two cases separately.

<u>Case 1: $a_1 + a_4 = a_2 + a_3 = a_5 = a_6$</u>

a. *Let's consider that the numbers satisfy the following property: $a_1 < a_2 < a_3 < a_4$.* Then, the number of ways we can choose a set of 4 questions that have points $a_1, a_2, a_3, a_4$, having this property is $n_{a1}n_{a2}n_{a3}n_{a4}$

   It is clear that the number of ways for choosing a set of 6 questions having this property can be calculated by

$$\sum_{a_5}\binom{n_{a_5}}{2}\sum_{a_1}\sum_{a_2>a_1} n_{a_1}n_{a_5-a_1}n_{a_2}n_{a_5-a_2}=\sum_{a_5}\binom{n_{a_5}}{2}\sum_{a_2} n_{a_2}n_{a_5-a_2}\sum_{a_1<a_2} n_{a_1}n_{a_5-a_1}=\sum_{a_5}\binom{n_{a_5}}{2}\sum_{a_2} n_{a_2}n_{a_5-a_2}P(a_2),$$

   where $P(a_2)=\sum_{a_1<a_2} n_{a_1}n_{a_5-a_1}$ . Note that $P(a_2)$ can be calculated using the sum for $a_2$ (that is $P(a_2)=P(a_2)+n_{a_1}n_{a_5-a_1}$ )

b. *Let's consider that our numbers satisfy the following property $a_1 < a_2 = a_3 < a_4$*: Then, the number of ways we can choose a set of 4 questions that have points $a_1, a_2, a_3, a_4$, satisfying this property is $n_{a_1}n_{a_4}\dfrac{n_{a2}(n_{a2}-1)}{2}$

   Similarly as in the previous case, the number of ways for choosing 6 questions having this property can be calculated by

$$\sum_{a_5}\binom{n_{a_5}}{2}\sum_{a_1}\sum_{a_2>a_1} n_{a_1}n_{a_5-a_1}\binom{n_{a_2}}{2}=\sum_{a_5}\binom{n_{a_5}}{2}\sum_{a_2}\binom{n_{a_2}}{2}\sum_{a_1<a_2} n_{a_1}n_{a_5-a_1}=\sum_{a_5}\binom{n_{a_5}}{2}\sum_{a_2}\binom{n_{a_2}}{2}P(a_2),$$

where $P(a_2)=\sum_{a_1<a_2} n_{a_1} n_{a_5-a_1}$ . Note that $P(a_2)$ can be calculated using the sum for $a_2$ (that is $P(a_2)=P(a_2)+n_{a_1} n_{a_5-a_1}$ )

c. *Let's consider that our numbers satisfy the following property: $a_1 = a_2 < a_3 = a_4$* : Then, the number of ways we can choose a set of 4 questions that have points $a_1$, $a_2$, $a_3$, $a_4$, satisfying this property is $\frac{n_{a1}(n_{a1}-1)}{2}\frac{n_{a3}(n_{a3}-1)}{2}$ .
Note that we can easily calculate the number of ways to choose 6 numbers having this property in $O(n^2)$.

d. *Finally, let's consider our numbers satisfy the following property: $a_1 = a_2 = a_3 = a_4$* . Then, the number of ways we can choose a set of 4 questions that have points $a_1$, $a_2$, $a_3$, $a_4$, satisfying this property is $\frac{n_{a1}(n_{a1}-1)(n_{a1}-2)(n_{a1}-3)}{3!}$ .
Note that we can easily calculate the number of ways to choose 6 numbers having this property in $O(n^2)$.

Case 2: $a_1 + a_2 + a_3 = a_4 = a_5 = a_6$

It is obvious that it is possible to calculate the number of combinations in this case in $O(n^3)$, but it is a a bit tricky to calculate it in $O(n^2)$. To do that we introduce a function dp2(k), the number of ways to choose 2 questions with a sum of points equal to k. This can be easy calculated in $O(n^2)$ time. Now we calculate the sum

$$A=\sum_{a_4}\binom{n_{a_4}}{3}\sum_{k} dp2(k) n_{a_4-k}$$

It is evident that this is not the number we are looking for, i.e. that some combinations are counted more than once. Therefore we will analyze how many times each combination is counted using the formula A.

Let's perform some case analysis:

Case 1: $a_1 < a_2 < a_3$. Such set of questions is counted exactly **3 times**, i.e. the first time is when $k= a_1 + a_2$ and $a_4-k= a_3$, the second time is when $k= a_1 + a_3$ and $a_4-k= a_2$ and the last time is when $k= a_2 + a_3$ and $a_4-k= a_1$.
Case 2: $a_1 < a_2 = a_3$ or $a_1 = a_2 < a_3$. In this case the combination appears twice in A, i.e. when $k= a_1 + a_2$ and $a_4-k= a_2=a_3$, and when $k= a_2 + a_3 = 2a_2$ and $a_4-k= a_1$. More precisely, for such set of questions we add the number

$$\binom{n_{a_4}}{3}\left\{n_{a_1}\binom{n_{a_2}}{2}+n_{a_1}n_{a_2}n_{a_2}\right\}=\binom{n_{a_4}}{3}n_{a_1}\left\{\binom{n_{a_2}}{2}+n_{a_2}n_{a_2}\right\}$$

While the actual number of ways to choose our set is
$\binom{n_{a_4}}{3}n_{a_1}\binom{n_{a_2}}{2}$.

But since it is difficult to distinguish the number of ways to choose our set when k is obtained as a sum of equal numbers, we want to count this number 3 times again. This can be achieved if we subtract the number $\binom{n_{a_4}}{3} n_{a_1} n_{a_2}$ , since

$$\binom{n_{a_4}}{3} n_{a_1} \left\{ \binom{n_{a_2}}{2} + n_{a_2} n_{a_2} \right\} - \binom{n_{a_4}}{3} n_{a_1} n_{a_2} = 3 \binom{n_{a_4}}{3} n_{a_1} \binom{n_{a_2}}{2}$$

Case 3: $a_1 = a_2 = a_3$. In this case the combination appears in A only once, when $k = 2a_1$ and $a_4 - k = a_1$, and the number that is added is $\binom{n_{a_4}}{3} n_{a_1} \binom{n_{a_1}}{2}$, which is not the correct one. Again it is in our best interest is to obtain correct number of ways to choose our set of 6 numbers, $3\binom{n_{a_4}}{3}\binom{n_{a_1}}{3}$, and it can be achieved if we subtract the number $\binom{n_{a_4}}{3} n_{a_1} \left(n_{a_1 - 1}\right)$ , since:

$$\binom{n_{a_4}}{3} n_{a_1} \binom{n_{a_1}}{2} - \binom{n_{a_4}}{3} n_{a_1} \left(n_{a_1 - 1}\right) = 3 \binom{n_{a_4}}{3} \binom{n_{a_1}}{3}$$

Therefore, in order to get the number of ways to choose our set in Case 2, we need to calculate:

$$\left( A - \sum_{a_4} \binom{n_{a_4}}{3} \sum_{a_1 < a_4} n_{a_1} n_{a_4 - 2a_1} - \sum_{a_4} \binom{n_{a_4}}{3} \sum_{a_{1=3a_4}} n_{a_1} \left(n_{a_1 - 1}\right) \right) / 3 .$$